Development of Framework with Exemplars Using Real-World Context to Teach Probability

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Abstract

Teaching of probability in mathematics classrooms has often been reduced to equipping students with a set of algorithmic procedures and formulas to handle similar questions. Being competent with the algorithmic procedures does not necessarily equate to the mastery of probability concepts. In this paper, we conducted a literature review and a study of the recent development of the Singapore Mathematics curriculum. Also tapping on the problem-solving nature of mathematics, we infused the use of real-world context in teaching probability to secondary school students. This is synchronous to the recent development in the Singapore mathematics education in which real-world context has been brought to the foreground into the school mathematics curriculum. In order to help students truly grasp probability concepts through the use of real-world context, we propose a TIDE (an acronym for 'Tackle students' misconception, Introduce probabilistic reasoning, Draw connections, and Encourage problem-solving') framework for designing a mathematics lesson on probability. An exemplar of a probability lesson using the Monty Hall Problem is presented in this paper.

Keywords: Probability education; Probabilistic reasoning; Real-world context; Problem-solving

Introduction

Background and Overview

Probability is a topic that is included in the mathematics curriculum for all students at both the upper and lower secondary levels in many countries, including Singapore. Both anecdotal evidence from the Singapore mathematics classrooms and research studies have shown that teachers frequently teach probability using a 'formula-based approach'. They tend to focus excessively on the formulas and algorithmic procedures in solving problems on probability without instilling in students some sense of probabilistic reasoning (e.g.,, Batanero & Diaz, 2012; Gal, 2005). Inspired by the recent trend of mathematics education in Singapore and the worldwide push for using real-world context for mathematics instruction, we explore the infusion of real-world context in the teaching of probability.

Aims and Rationale

In this paper, we propose a framework (TIDE, an acronym for Tackle, Introduce, Draw and Encourage) for mathematics educators to design and enact an introductory probability lesson through the use of real-world context. To develop students' probabilistic reasoning in order to have a deeper understanding of the concept of probability is underlying the objective of the framework that we propose. We also demonstrate the application of the TIDE framework in designing an introductory lesson on Probability using the famous Monty Hall Problem (e.g., Ho, 2008; pp. 179).

Literature Review

Skills Mastery versus Conceptual Understanding in Mathematics

Mathematics educators from many countries have reported that mathematics lessons and school-based assessments typically focus on developing and assessing students' ability to execute mathematics algorithms or recall and apply mathematics formulas (e.g., Bergsten, 2002; Tan, 2011). Based on a recent research conducted in the secondary schools in Singapore, Wong and Kaur (2015) reported that more than 50% of all the mathematics assessments analysed in their study heavily focused on assessing students' mastery of routine procedures. Many of the assessment items in the school-based examinations focused heavily on the skills aspect of the students, usually at the expense of conceptual understanding. As it is a well-known fact that assessment drives the way that lessons are delivered, this observation could be suggestive that Singapore mathematics teachers might have excessively focused on formulas and procedures during their classroom instruction.

It is acknowledged that learning mathematics is hierarchical in nature in the sense that the more advanced skills are built on the elementary ones. However, being able to solve a mathematics question at the various hierarchical levels of skills does not necessarily mean that the students have understood the underlying meaning of the various algorithmic procedures, the concepts behind the solution of the mathematics question, or even the question itself (Schoenfeld, 1988). In short, task mastery does not necessarily equate to an understanding of the mathematics concepts.

In addressing the need to strive for deep understanding in mathematics, some researchers have advocated the use of real-world applications in the teaching of Mathematics after realizing its benefits. For example, Flannelly (2014) used real-world contexts in teaching her Algebra I class in a high school. Comparing with the other group of students who were taught using traditional instructional material without emphasizing on real-world contexts, and she reported that this group of students displayed a better conceptual understanding of the topic. The written work submitted by her students was of much better standard compared to the work of the students who had gone through the traditional instructional approach. She also observed that her students' interest in the subject was very much enhanced. Another study carried out by Çetin (2004) showed that the use of real-world context not only brought about greater motivation amongst his students, but also a better understanding of mathematical concepts.

The underlying benefits of the use of real-world context include students are assisted to see the value in the content that they are learning also autonomous learning is encouraged (Ryan & Deci, 2000). This will in turn bring the students a step closer to attaining intrinsic motivation where they are driven by not external factors, but themselves (Niemiec & Ryan, 2009). The active learning from students and a deep information processing resulting from the use of real-world context in mathematics instruction would be vital in helping students learn (Niemiec & Ryan, 2009). Therefore, it seems that using real-world context can indeed promote conceptual understanding of mathematics. Some educators are advocating striking between a balance between mathematics content and the use of real-world context to achieve the maximal result in students' learning of mathematics (e.g., Harvey & Averill, 2012).

A study on using real-world context for mathematics instruction was conducted by Tsang and Shahrill (2015) specifically on the teaching of probability. The study showed that their students were more engaged and participative when they were tasked with a probability problem which was situated in a real-world context. The teachers involved in the study also noticed that, as a consequence of using real-world context in teaching probability, the students started to ask more 'good questions'. This could be an indication that the students were actively thinking when they were engaged with problems in a real-world context.

Issues and Challenges in the Teaching of Probability

Batanero, Chernoff, Engel, Lee, and Sánchez (2016) asserted that it is common that teachers only focus on presenting the different probability concepts and the way it is applied in questions, but failing to bring across the different meanings of probability. This approach of teaching probability among teachers prevents students from truly understanding the concepts behind probability. It is thus not surprising that students might not be able to articulate what the specific probability concept represents in the real-world. Some common concepts that students usually have difficulty include randomness and sample space (Sánchez, Garcia-Garcia & Mercado, 2018).

Probability can be understood as a distinct approach for people to ponder and explain various random events that they experience in the real world. In this sense, probabilistic reasoning can be understood as how a person judges and weighs various possibilities for decision-making (Batanero et al, 2016). However, anecdotal evidence and our collective classroom experience show that most students seldom engage in probabilistic reasoning. The students tend to tilt towards deterministic thinking instead. Deterministic thinking refers to how a person focuses on the specific event itself rather than *all the possible events* (Batanero, Henry, & Parzysz , 2005) or feels that only a certain outcome would be obtained (Sánchez et al., 2018). Sánchez et al. (2018) have also found that common misconceptions about probability among students lead them to deterministic thinking. Some of these misconceptions about probability include 'representativeness' and 'human control' (Ang & Shahrill, 2014), which is elaborated in the next subsection in this paper.

Nguyen (2015) argued that if probability is learned purely as algorithms and procedures, learners would not be able to attain much depth into the concepts or experience revelations for themselves. He also mentioned how probabilistic reasoning could bring about developments in intuition, which in turn led to a deeper understanding of probability. Nguyen (2015) pointed out that getting students to see situations as problems to be solved instead of a procedure to follow would also allow the progression of probabilistic reasoning. Therefore, instead of merely using skills, formulas or algorithmic procedures that they have learnt previously, teachers could engage their students to reason “probabilistically” to solve a new probability problem given to them. Note that he did not totally discredit the usefulness of formulas as he felt that formulas could actually help to integrate the various concepts together as well as gave students an alternative method to make sense of the concepts acquired.

Some researchers have argued that students should be presented with opportunities to utilize their intuition to aid in their understanding and interpreting of probability (e.g., Sharma, 2015). An example of how a probabilistic concept can be linked to intuition is the concept of independence (Batanero et al., 2005). Batanero et al. (2005) elaborated that the intuitive component would be when one feels like there is no indication to suggest that one event can influence another and the probabilistic translation of this idea is expressed via the multiplication rule.

To provide opportunities for utilizing students’ intuition in learning probability, Sharma (2015) proposed the inclusion of activities in which students have the opportunity to perform empirical investigations to check against their intuition. She suggested that, instead of directly introducing the definition of probability using a formula, students should be presented with a problem as well as tasked to predict, observe, and then discuss their intuitive answers. Her rationale was that the prediction component would interest students and spark their curiosity as they are likely to be excited to want to know the correctness of their answers based on their intuitive reasoning. In addition, the discussion about their answers would allow students to be aware of what other students think and thus evaluate their predictions as well as find out whether their initial intuitions were accurate. In short, Sharma (2015) believed that getting students to “explore, conjecture, reflect on any discrepancies they observe, evaluate and explain their reasoning” could be instrumental in developing students’ probabilistic thinking. This echoes what Jones and Thornton (2005) suggested with regard to learning probability. He stated that learning of probability first starts with having individual’s opinions and then comparing it with the empirical situation (frequentist approach) as well as the theoretical models (classical approach).

However, researchers also cautioned that merely engaging students with such investigative activities does not necessarily result in meaningful learning (Nilsson, Eckert, & Pratt, 2018). Nilsson et al. (2018) reasoned this with the existence of misconceptions with regard to empirical investigations that can distract students (which will be elaborated in the following subsection). If the misconceptions are left unaddressed, students may still resort to deterministic thinking instead of probabilistic reasoning in solving probability questions.

Misconceptions about the Concept of Chance and Sample Space

Jones and Thornton (2005) mentioned that learning comes by first getting personal opinions about a random situation, followed by comparing the empirical situation as well as conjectured theoretical model, and using the comparisons made, generalizations can be formed. They illustrated with the example that people tend to believe that in a sequence of a random die, the sequence 1,2,3,4,5,6 is less likely than 2,5,1,6,4,3 as the sequence with less “regularity” seems more probable. This same misconception is also discussed by Hope and Kelly (1983). Hope and Kelly (1983) mentioned that people tend to treat seemingly ‘unusual’ events as less probable, which in this case, regularity is unusual. Thus, Jones and Thornton (2005) suggested that both classical and frequentist approaches need to be taught together with the concept of chance.

Hope and Kelly (1983) also pointed out the struggle to isolate the prediction of an independent event with a similar event that has occurred before. They partially attributed this struggle to sayings like ‘lightning never strikes twice in the same place’ which suggests that the occurrence of one event is always affected by the occurrence of previous such event. The same difficulty is further elaborated by Ang and Shahrill (2014), who referred to this misconception as ‘representativeness’. They described this misconception as when students “think that samples which correspond to the population distribution are more probable than samples which do not”. To illustrate this, they used the case of a coin toss. Students struggling with representativeness would feel that obtaining a series of coin tosses that have almost equal number of heads and tails would be more probable than one that has an unbalanced number (i.e., many more heads than tails). This reasoning thus leads students to think that heads would be more likely to appear than tails if the same coin has already produced 4 tails in a row when in fact, that is not true.

Ang and Shahrill (2014) described another misconception among students that an outcome is beyond their control, that is, dependent on external forces (inclusive of divine intervention), or based on their own experience of similar events. One of the respondents in their study commented that the rationale for her answer was that ‘she never tosses four heads in a row for a coin toss’, suggesting this belief misconception. Jones and Thornton (2005) proposed that the emphasis on sample space can help students see the possible outcomes and thus prevent students from conforming to deterministic thinking. Therefore, helping students address their misconceptions with regard to the concept of sample space would also encourage growth in probabilistic thinking.

However, learning how to obtain all possible outcomes would only be of value to the students if they appreciate the importance of sample space in affecting the probability of the event (Nilsson et al., 2018). It was shown by Nilsson et al. (2018) that when students were asked to consider the probabilities of the sum of two non-cubical dies, many were focusing on the material features of the dies, instead of considering the possible numbers obtained by the two dice.

Another possible misconception mentioned in the study by Ang and Shahrill (2014) is ‘equiprobability bias’ where students tend to believe that random events are ‘equally probable by nature’. This misconception also highlights students’ lack of consideration for sample space. Moreover, in the research by Nilsson et al (2018), they found out that students feel that they ‘can control the outcome of a throw’. This misconception, ‘human control’ is also mentioned by Ang and Shahrill (2014) where students think that outcomes are affected how one ‘throws or handles’ the coin or die involved.

Methodology: Teaching Probability Using the TIDE Framework

This article employs qualitative ‘multiple case analysis’ approach to exemplify how the framework developed by the authors can be applied in teaching probability at secondary levels using real-world contexts. In this section TIDE framework for designing a lesson on probability will be presented. Fundamentally, the TIDE framework proposes infusing problem-solving approach within a lesson that emphasizes probabilistic reasoning. The problem-based approach in the TIDE framework is aligned to the problem-solving approach towards mathematics. Rather than perceiving it as an innovation, the framework is more accurately perceived as a re-interpretation of the problem-solving framework as discussed in Toh, Quek, and Tay (2008).

Teaching Probability with Help from Problem-Solving

Based on the literature review, we first identified the areas of focus for mathematics educators for designing a lesson to teach probability. Although the analysis carried out by Yang and Sianturi (2019) shows that students are expected to make connections when performing a mathematics task, there is still a heavy emphasis on formulas and algorithms. Therefore, in developing our approach of teaching probability, we seek that probability should not be taught in such a manner that the concepts are reduced to formulas and procedures. Hence, we envision an approach of teaching probability such that students have not have just achieved procedural fluency in probability based on the formulas and the algorithms, but also understand the rationale as well as concepts behind each step when the concepts are applied, and, equally important is developing students' probabilistic reasoning.

To prevent mathematics instruction from reducing mathematics to simply procedures and formulas, we believe that a problem-based approach is critical in designing a probability lesson. According to Toh, Quek, and Tay (2008), mathematics instruction using a problem-based approach provides students with the opportunity to 'think like a mathematician' by allowing them to go through the processes of mathematical problem-solving, instead of merely trying to apply various formulas or series of steps that they can recall. Going through the problem-solving processes, a student will undergo the various steps of Polya (1945), that is, attempt to apply heuristics to both *understand* and *solve* the problems. To take problem-solving one step further by infusing this problem-solving approach with real-world context, we propose an approach that makes use of mathematical problem-solving to tackle problems in real-world context.

Pertaining to teaching probability, it is clear from the literature review presented in the preceding section that probabilistic reasoning is what we want to nurture in the students in addition to the usual deterministic thinking. However, it has been shown that most students tend to rely on deterministic thinking instead. This over-reliance on deterministic thinking could be attributed to their common misconceptions of chance or probability. Therefore, we have to first tackle students' common misconceptions on chance by moving them away from mere deterministic thinking in learning probability.

Another important area in getting students to grasp probabilistic reasoning is to help them appreciate how their own intuition plays a crucial role in developing their probabilistic reasoning. Intuition cannot only help students be aware of their own misconceptions (if they realise that their 'gut feeling' is 'wrong'), it can also help them make sense of various probability concepts (if they realise that their 'gut feeling' is 'accurate').

Thus, we propose the TIDE framework. It consists of 4 suggestions that should be addressed in order to teach probability effectively.

T: Tackle students' misconception

I: Introduce probabilistic reasoning

D: Draw connections

E: Encourage problem-solving

The summary of the model is shown in the following Figure 1.

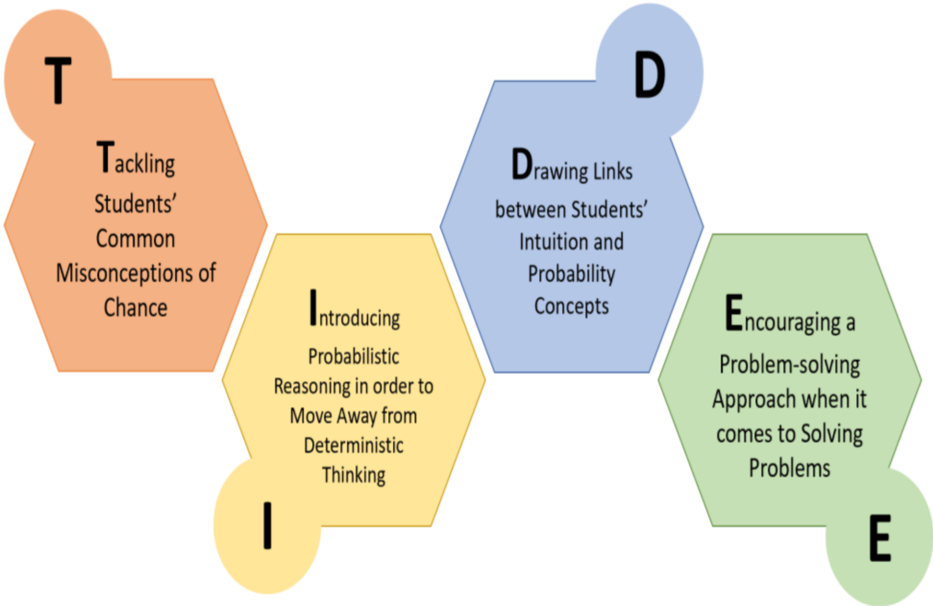

Figure 1. The TIDE framework on teaching probability.

Designing a Probability Lesson using the TIDE Framework

Based on the TIDE framework, we crafted an activity plan on the introductory lesson of probability as an illustration (It is instructional for one to design the subsequent lessons on probability using the TIDE framework). Appreciating the benefits of real-world contexts in mathematics education, the overarching element of this whole activity plan is embedding real-world context into the mathematics activity The following are two main features observed to the activity plan, namely The Hook and The Game anchoring on TIDE framework with discussions based on the authors' experience in the implementation of piloted activities among small groups of samples. An initial attempt to use this approach of teaching probability to lower secondary school students was piloted by the second author in his engagement with students for mathematics enrichment classes in several mainstream secondary schools. However, a full-scale implementation or an empirical study of the approach proposed in the TIDE framework has yet to be carried out.

(1) The Hook

The Hook is an attempt to gain student's attention by providing thought-provoking questions regarding probability. This creates the opportunity to Tackle students' misconception about probability and Introduce probabilistic reasoning to the students ("TI" component of the TIDE framework). The thought-provoking questions are crafted to elicit common misconceptions among students through allowing them to share their opinion. This enables students to be aware of their own misconceptions as well as teachers to be aware of their students' understanding (and misunderstanding) about probability.

Tackling students' misconception using questioning technique. We propose that two introductory questions which target specific misconceptions are posed to the students as elaborated below.

Question 1

If you could pick a 4-digit lottery number, what number would you pick?

Question 2

In a game of flipping a coin, four heads have already turned up. What should you bet on the next flip?

Question 1 tackles the common misconception of chance "unusual" events (an "unusual" 4-digit number, for example, 1111 in the lottery) are thought to be less probable or less likely to win than the "usual" ones (e.g., 2365). The students should be struck to realize that in fact every of the ten thousand 4-digit numbers (from 0000 to 9999), in fact has an equal chance of winning. This intuitive question should lead students to the foundation of sample space and the notion of "equally likely" in the definition of probability to be introduced in the latter lessons on probability.

Question 2 tackles the common misconception about dependent and independent events. The common misconception among students in Question 2 is that the outcome of an event is almost always affected by the outcomes of the previous events, or known as the "representativeness misconception". This is an attempt to move students away from the "gambler's mentality", that is, all events are affected by previous occurrence or non-occurrence of events. This prepares the students for the probabilistic notions of dependence and independence of events.

Introducing probabilistic reasoning through teacher scaffolding. Teachers should allow their students at least one minute to think about what they would personally do before they open up the floor for them to share their thoughts and opinions. The teacher could encourage students to explain the reason behind their choice rather than just accept an answer which might be randomly given. A discussion can also be facilitated. It should be noted that the objective of the discussion segment of this lesson is to arouse cognitive conflict in the students so as to provide an opportunity for them to question their intuition prior to learning probability formally, thereby accommodation and assimilation can occur in learning the new concept of probability.

(2) The Game

The objective of the game introduced in this lesson is to allow students to carry out empirical investigations on chances.

Drawing connection with real world context. This gives the students the opportunity to Draw connections between the game, the hook and the underlying probabilistic reasoning, followed by the teacher Encouraging them to use problem-solving approach to solve the problem of the games ("DE" part of the TIDE framework). Both Sharma (2015) and Jones and Thornton (2005) advocated this approach to allow for students to compare their intuition versus the concepts of probability, hence enabling them to develop probabilistic reasoning in order for learning to take place. This component of the lesson provides students with the experience to challenge their preconceptions about chance and probability: make predictions, carrying out an empirical investigation, obtaining data and drawing conclusions via making comparisons. In this way, the students are given the opportunity to apply their problem-solving strategies in solving the problem of the game.

Facilitating discussion among students engages them actively in constructing the classroom discourse. Thus, the typical scenario in which teachers merely show or explain the solutions and answers, which can hardly engage the students, should be avoided. To facilitate student discussion effectively, the strategies advocated by Nilsson et al (2018), as well as Toh, Quek, Leong, Dindyal, and Tay (2011) are introduced. Teachers should be cautious not to be overly judgemental. Reduced evaluation by the teacher will likely lead to less pressure for students to openly share their ideas. However, a totally student-led discussion without teachers' input is unlikely to lead students to any meaningful learning. Thus, the teacher as a facilitator needs to strike a balance between the two extremes.

Encouraging problem-solving using The Monty Hall Problem. The Monty Hall Problem was popularized between 1963 and 1990 due to Monty Hall's television show, "Let's make a deal" (Bennett, 2018). This problem involves winning the grand prize by picking the correct door. First, the player gets to pick one out of three doors where one hides the prize but the other two hide the goats. Once the choice has been made, the host (who knows where the prize is) would then open one out of the remaining two doors to reveal a goat. Players then get the option to stick with their initial choice, or to switch the door. As reported in Bennett, the controversy over the correct decision only arose when Marilyn vos Savant, Guinness Book record holder of highest IQ, pointed out that one should choose to switch. This sparked many reactions where 92% of the letters received claimed that she was wrong and 65% of the letters from universities begged to differ too (Bennett, 2018). One reason why this problem is so famous is also due to how "even the finest and best-trained minds get trapped" (Dupont & Durham, 2018). Numerous studies have shown that the vast majority of around 4.5 to 21% (e.g., Bown, Read, & Summers, 2003; Friedman 1998; Granberg & Brown 1995) chose to stick with their initial choice when in fact, choosing to switch would double one's chances of winning.

The probability concept in this problem involves conditional probability (Saenen, Dooren, & Onghena, 2014), which is about a (reduced) sample space (Bennett, 2018). However, it is likely that most students view the dilemma of whether to switch or stay with the original choice as independent of the initial choice made by the player. This then leads to them incorrectly believing that there is no difference in the chance of winning whether to switch or to stay with the original choice, as both choices yield equal probability of winning (Saenen et al., 2014). Most probably they fail to see the dependence between initial choice and the choice to switch or stay because they do not recognise that the information provided by the host through opening one door with the goat is actually not random (Bennett, 2018). This incorrect reasoning has also been associated with the common misconception of equiprobability bias (Saenen, Heyvaert, Dooren, & Onghena, 2015), which has also been mentioned in the literature review in the preceding section. However, from systematically listing all possible scenarios (i.e. listing all the possible events in the sample space), one can easily appreciate that a switch of the choice will increase one's probability of winning (Bennett, 2018).

In addition, there are psychological (non-probability related) factors that can influence one's decision too. One such factor that Bennett (2018) pointed out is about an individual's commitment and regret. Gilovich, Medvec, and Chen (1995) showed that under decision-making situation, a person who switched his or her choice and yet lost a game will 'experience more regret and psychological pain' compared to those that stayed and lost. Many people might then choose to stick with their choice as they do not want to feel such regret.

The Monty Hall Problem would definitely intrigue the vast majority of students, regardless of their background and culture. Not only the problem is set in the context of a game about decision-making, its solution is counter-intuitive to many students and adults. Thus, this is a good problem to be used to challenge students' probabilistic thinking. This is also a proposed problem for teaching probability suggested by Ho (2008), although she did not provide details how a lesson involving the Monty Hall problem could be enacted. The Monty Hall Problem can also lay the foundation in addressing students' misconception of equiprobability, independence and conditional probability, as well as sample space in the subsequent lessons of probability.

Facilitating the Game

Based on the experience of implementing TIDE framework, the following aspects are identified as important features/steps to facilitate the game probability.

Forming of hypothesis. After explaining the Monty Hall problem, students are to spend at least a minute to make a choice, as well as to write down the reasoning behind their choice. As the objective here is to encourage students to experience learning from empirical investigations, the teacher will not discuss the correct answer here. The students' reasoning is also not evaluated at this juncture.

Investigation. A frequentist approach can next be used for students to experiment the game multiple times to obtain an estimation of the probability. In other words, the students can carry out the physical simulation of the game to gather their own sample data. They can be divided into pairs in which they switch between the role of the 'host' and the 'player' in order to manually record the number of times of winning the game versus the decisions they have made. Similar to the actual game, only the host knows the location of the prize. By using three sheets of paper representing the three doors, the host reveals one of the doors which has a goat that is not picked by the player, while the player can choose if they want to change their choice of door. An example of how students can record their sample data and obtain their estimation is shown below.

Choice: Switch

Trial number	Win (Y/N)
1	
2	
…	
N_1	

Total wins = Number of "Y"s = W_1

Choice: Do not switch

Trial number	Win (Y/N)
1	
2	
…	
N_2	

Total wins = Number of "Y"s = W_2

Probability of winning if choice is to switch = $W_1 / (N_1 + N_2)$

Probability of winning if choice is to not switch = $W_2 / (N_1 + N_2)$

Drawing conclusions. With data collected, the students would evaluate their initial choice of answer. This provides the learning experience required for students to switch from deterministic to probabilistic thinking. Due to limited class time, the number of individual trials completed would still be relatively small hence, the teacher should use the online simulator to carry out a large number of trials (Appendix A). To obtain an even larger number of trials, the teacher could also sum up all the trials and the number of wins for each decision (to switch or not to switch) made completed by all the students. Comparisons between frequentist and classical approach can also be made here. The key message for students in this introductory lesson is an initiation of the students into probabilistic thinking, for example, the probabilistic value of 2/3 does not necessarily mean that one would definitely deterministically obtain two successes out of a total of three attempts.

Using Problem-Solving to check the correctness of their answers. The Monty Hall Problem can also be solved by using problem-solving approach without an explicit use of probabilistic reasoning. This could be solved by listing out the cases and comparing between the decision to switch the choice and not to switch. To this end, the students are exposed to alternative approaches to solve a problem on probability and, more importantly, to use the usual problem-solving approach to "validate" their probabilistic approach. In concluding the lesson, the students could be invited to reflect on their intuition on probability versus the results from the empirical data, and the validity of their choice in the light of mathematical problem-solving. This is an opportunity for teachers to introduce the concept of sample spaces in the introduction of probability that will come in handy when the definition of probability is introduced in the subsequent lessons.

Discussion

Use of the TIDE framework

In providing a list of strategies on teaching probability, Ho (2008) suggested that the concepts of probability should be introduced through activities and experimentation (pp. 178 – 179). It was also suggested that students must be encouraged to analyse probabilistic statements critically. The TIDE framework proposed in this paper in fact is a materialization of the suggested teaching approaches by Ho (2008). The use of the hook questions serve to arouse cognitive dissonance among the students in order for them to debunk their intuitive ideas about chance and probability.

The game we chose in this lesson design (the Monty Hall problem) can be seen as a problem in the real-world context. A problem in the real-world context that allows students to make decision empowers them for their own learning (Toh, 2010). This is likely to arouse their interest and keep them engaged in the mathematical tasks.

As the Monty Hall Problem is non-routine to most students, they will be diverted from their usual habit of simply thinking about formulas and algorithmic procedures to apply. Thus, they will be encouraged (E) to use problem-solving approach, which they have been familiar. Next, the stages of problem-solving throughout this problem is outlined.

Forming of hypothesis. This component, which is one of the important heuristics of mathematical problem-solving, requires students to make use of their intuition or prior knowledge to make a hypothesis about the best decision they will take. Getting students to make a guess would also spark their curiosity, as they would be keen to know if they are right.

Investigation. This hands-on section further amplifies the real-world context as the students actually carry out the activity in real life. Students also get to experience the frequentist approach to probability. According to Tyler (Madeus & Stufflebeam, 1989), the process of learning is as important as the product of learning. Thus, learning experience in the form of investigative task through collecting empirical data, is crucial for learning probability.

Drawing conclusions. Using probabilistic reasoning is once again carried out here when the teacher points out the difference between the two. The evaluation of their own reasoning and attempting to make sense of the probability obtained via the frequentist approach, that is, the carrying out of the physical simulation of the game or to use a computer simulation, allows students to link their intuition to the probability concepts. Learning about the main gist of probability would occur here where students get to observe and compare their predictions, empirical situation (frequentist approach) and theoretical models (classical approach), allowing them to draw various generalisations.

Conclusion

In this paper, we have presented the TIDE framework for teaching probability in the high school, based on literature review conducted in mathematics education on the teaching of probability, and mathematics teaching using real-world context. In Toh et al. (2011), a problem named the 'Phoney Russian Roulette' (p. 82) has been introduced through the classical problem solving approach. Readers are invited to design a lesson using the TIDE framework presented in this paper to introduce conditional probability through the Phoney Russian Roulette problem.

In short, the TIDE framework is an attempt to infuse real-world context, mathematical problem-solving and probabilistic reasoning in the teaching of probability. We have demonstrated the use of TIDE framework to design the first introductory lesson on probability. The TIDE framework can also be used to design the subsequent lessons on probability. Due to the constraint of time, we were not able to trial the lesson in an authentic mathematics classroom. However, we hope that this will spur an interest into infusing real-world context in the teaching of probability.

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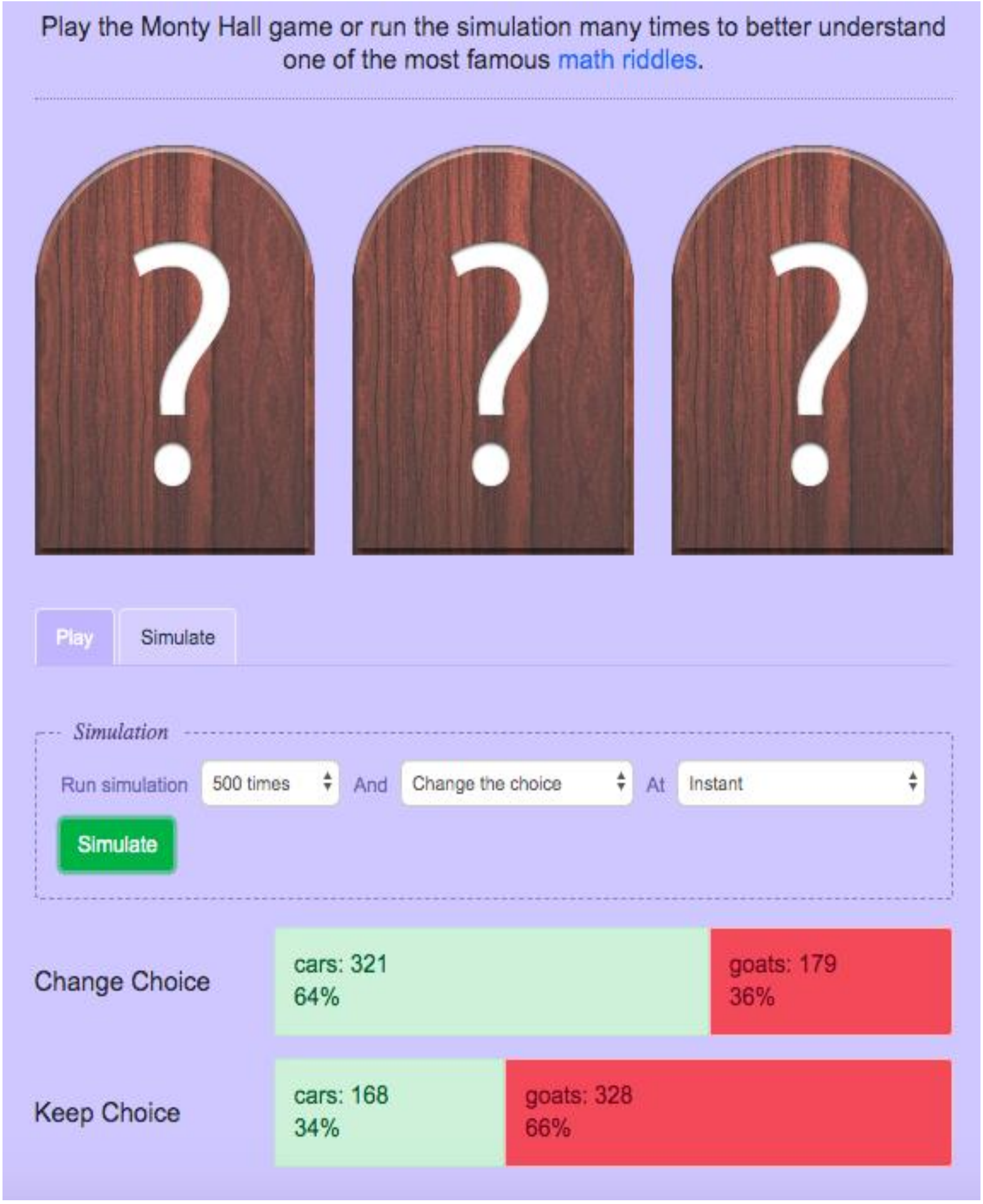
APPENDIX



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